Nature Academy

Mathematical methods in Physics

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Exact Differential Equations

Exact Differential Equations

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Exact D·E

Eg:- $\cos(x+y)\,dx + (3y^2+2y+\cos(x+y))\,dy = 0$

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(\cos(x+y))dx = -\sin(x+y)$$

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left[(3y^2+2y+\cos(x+y)\right]dy = -\sin(x+y)$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

How to Solve it :-

Working Rule :-

1) Write the given D.E in the form

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

2) find $\frac{\partial M}{\partial y}$ & $\frac{\partial N}{\partial x}$ $\Rightarrow$ if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

3) $\int M\,dx + \int$ (terms of N not containing x) $dy = c$

Eg:- $(x^2-4xy-2y^2)\,dx + (y^2-4xy-2x^2)\,dy = 0$

$$M \, dx + N \, dy = 0$$

$$M = x^2-4xy-2y^2 \qquad N = y^2-4xy-2x^2$$

$$\frac{\partial M}{\partial y} = 0-4x-4y \qquad \frac{\partial N}{\partial x} = 0-4y-4x$$

$$= -4x-4y \qquad = -4y-4x$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ $\therefore$ this a exact D.E.

$\therefore \int M\,dx + \int (\text{terms of } N \text{ not containing } x)\,dy = c$

$$\int (x^2-4xy-2y^2)\,dx + \int (y^2)\,dy = c$$

$$\frac{x^3}{3} - \cancelto{2}{4}y \cdot \frac{x^2}{\cancel{2}} - 2y^2x + \frac{y^3}{3} = c$$

$$\frac{x^3}{3} - 2yx^2 - 2y^2x + \frac{y^3}{3} = c$$

Equations Reducible to the Exact Form

If $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

Multiply a I·F with the given D·E.

1) Homogeneous D·E

$M\,dx + N\,dy = 0$ in x & y then

$$IF = \frac{1}{Mx + Ny}$$

Homogeneous D·E is in the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$

same order
same degree of each term.

Eg:- $\underbrace{(x^3+y^3)}_{M}dx \underbrace{- xy^2}_{N} dy = 0$

$IF = \dfrac{1}{Mx+Ny}$

$M = x^3+y^3 \qquad N = -xy^2$

$\dfrac{\partial M}{\partial y} = 3y^2 \neq \dfrac{\partial N}{\partial x} = -xy^2$

$IF = \dfrac{1}{(x^3+y^3)x + (-xy^2)y} = \dfrac{1}{x^4 + \cancel{xy^3} - \cancel{xy^3}} = \dfrac{1}{x^4}$

$x^r y \quad IF = \dfrac{1}{x^4}$

$\dfrac{1}{x^4}(x^3+y^3)dx - \dfrac{1}{x^{\cancel{4}3}}(\cancel{x}y^2)\,dy = 0$

$\left(\dfrac{1}{x} + \dfrac{y^3}{x^4}\right)dx - \dfrac{y^2}{x^3}\,dy = 0$

$$\left(\frac{1}{x}+\frac{y^3}{x^4}\right)dx-\frac{y^2}{x^3}\,dy=0$$

$$\frac{\partial M}{\partial y}=\frac{3y^2}{x^4} \quad = \quad \frac{\partial N}{\partial x}=\frac{3y^2}{x^4}$$

$$\int M\,dx+\int N\,dx=c$$

$$\int\left(\frac{1}{x}+\frac{y^3}{x^4}\right)dx+\underbrace{\int 0\,dx}_{0}=c$$

$$\ln x+\int\frac{y^3}{x^4}\,dx=0$$

$$\ln x+y^3\cdot\frac{x^{-3}}{-3}=0 \implies \ln x-\frac{y^3}{3x^3}=c$$

$$\frac{\partial}{\partial x}\left(\frac{y^2}{x^3}\right)$$

$$=y^2\frac{\partial}{\partial x}\left(\frac{1}{x^3}\right)$$

$$=y^2\frac{\partial}{\partial x}\left(x^{-3}\right)$$

$$=y^2\cdot\left(-3x^{-4}\right)$$

$$=\frac{3y^2}{x^4}$$

2) If the equation is of the form

$$\underline{f_1(x,y)\,y}\,dx + \underline{f_2(x,y)\,x}\,dy = 0$$

then take $IF = \dfrac{1}{Mx - Ny}$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

Eg:- $(1-yx)\,y\,dx + (1+xy)\,x\,dy = 0$

$$\underbrace{(y-y^2x)}_{M}\,dx + \underbrace{(x+x^2y)}_{N}\,dy = 0.$$

$$\frac{\partial M}{\partial y} = 1-2yx \qquad \frac{\partial N}{\partial x} = 1+2xy$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\begin{aligned} Mx - Ny &= (y-y^2x)x - (x+x^2y)y \\ &= \cancel{xy} - x^2y^2 - \cancel{xy} - x^2y^2 \\ &= -2x^2y^2 \end{aligned}$$

$$IF = -\frac{1}{2x^2y^2}$$

$x^{n}y$ IF $\Rightarrow$

$$\frac{1}{x^2y^2}\left[(y-y^2x)dx + (x+x^2y)\,dy\right] = 0$$

$$\underbrace{\left(\frac{1}{x^2y} - \frac{1}{x}\right)}_{M'}dx + \underbrace{\left(\frac{1}{xy^2} + \frac{1}{y}\right)}_{N'}dy = 0$$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$$

$$\int\left(\frac{1}{x^2y} - \frac{1}{x}\right)dx + \int \frac{1}{y}\,dy = c$$

$$-\frac{1}{xy} - \ln x + \ln y = c$$

$$\int\left(\frac{1}{x^2y} - \frac{1}{x}\right)dx$$

$$= \frac{-1}{xy} - \ln x$$

$$\int \frac{1}{x^2y}\,dx = \frac{1}{y}\int x^{-2}dx$$

$$= \frac{1}{y}\,\frac{x^{-2+1}}{-2+1}$$

$$= -\frac{1}{xy}$$

3) If the equation is $Mdx + N\,dy = 0$

a) $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x)$ $\quad$ $\underline{IF = e^{\int f(x)\,dx}}$.

$(2x\log x - xy)\,dy + 2y\,dx = 0$.

b) $\frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = f(y)$ $\quad$ $\underline{IF = e^{\int f(y)\,dy}}$

$(y^4 + 2y)\,dy + (xy^3 - 2y^4 - 4x)\,dx = 0$

a) $(2x\log x - xy)\,dy + 2y\,dx = 0$

$$N\,dy + M\,dx = 0$$

$$M = 2y \qquad N = 2x\log x - xy$$

$$u \cdot v$$

$$\frac{\partial M}{\partial y} = 2 \qquad \frac{\partial N}{\partial x} = 2\cancel{x}\cdot\frac{1}{\cancel{x}} + 2\log x - y$$

$$\frac{\partial N}{\partial x} = 2(1+\log x) - y$$

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{2x\log x - xy}\left(2 - 2(1-\log x) + y\right)$$

$$= \frac{\cancel{2}\left[\cancel{1}-\cancel{1} + \cancel{\log x + y/2}\right]}{-\cancel{2}x\left[\cancel{\log x + y/2}\right]} = -\frac{1}{x}$$

$$= f(x)$$

$$IF = e^{\int \frac{1}{x} dx}$$

$$= e^{-\log x} = \frac{1}{x}$$

$$e^{-\log x} = e^{\log x^{-1}}$$

$$= \frac{1}{x}$$

$$\frac{1}{x}\left[(2x\log x - xy)dy\right] + \frac{1}{x}(2y)dx = 0$$

$$\underbrace{2\log x - y}_{N}\, dy + \underbrace{\frac{2y}{x}}_{M}\, dx = 0.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\int \frac{2y}{x} dx + \int (-y) dy = c$$

$$2y\log x + \left(-\frac{y^2}{2}\right) = c \Rightarrow 2y\log x - \frac{y^2}{2} = c$$

Q:- $(\underbrace{xy^2 + \lambda x^2 y}) dx + (\underbrace{(x+y) x^2}) dy = 0$ is an exact D.E. find λ.

a) 1 b) 2 c) 3 d) 4

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial}{\partial y}(xy^2 + \lambda x^2 y) = \frac{\partial}{\partial x}\left[(x+y)x^2\right]$$

$$2xy + \lambda x^2 = \frac{\partial}{\partial x}\left[x^3 + x^2 y\right]$$

$$\cancel{2xy} + \lambda x^2 = 3\cancel{x^2} + \cancel{2xy}$$

$$\lambda = 3$$

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